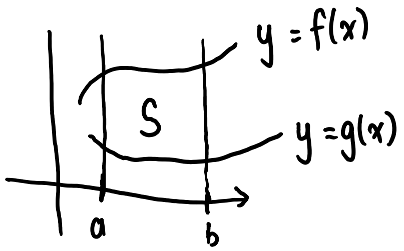
Lecture 26

Wednesday, November 9, 2016 9:26 AM

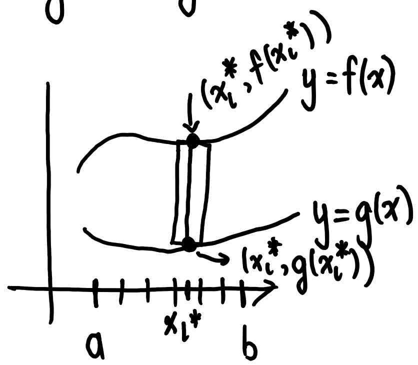
Area Between Curves

Consider a region S that lies betn two curve $y = f(x)$, $y = g(x)$ and betn vertical lines $x = a$ and $x = b$.
Let f, g be continuous func and $f(x) \geq g(x)$ for all x in $[a,b]$.



GOAL
Compute area A of region S

Just as we did in prvs class, divide $[a,b]$ into n-strips of equal width Δx, and then approximate area of each strip using rectangles.



The i^{th} rectangle has height $f(x_i^*) - g(x_i^*)$ and width Δx

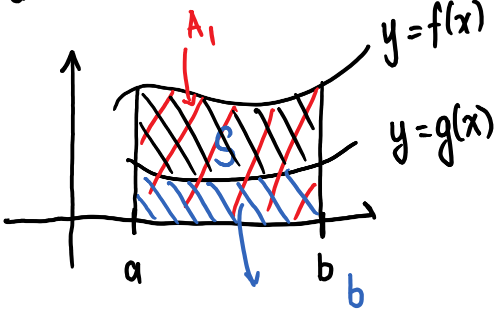
$$A_n = \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \cdot \Delta x$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} \left[f(x_i^*) - g(x_i^*) \right] \cdot \Delta x$$

The area of region S is

$$A = \int_a^b [f(x) - g(x)]\, dx .$$

Remarks

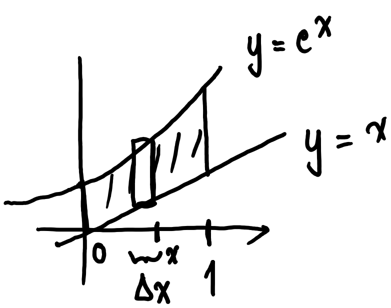


$$A_1 = \int_a^b f(x)\, dx , \qquad A_2 = \int_a^b g(x)\, dx$$

$$A = A_1 - A_2 = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$$

$$= \int_a^b [f(x) - g(x)]\, dx$$

Ex Find the area of the region bounded by $y = e^x$, $y = x$ and vertical line $x = 0$ and $x = 1$.



The ht of the approximating rectangle is

$$y_T - y_B = e^x - x$$

$$A = \int_0^1 (e^x - x)\, dx = \left[e^x - \frac{x^2}{2} \right]^1$$

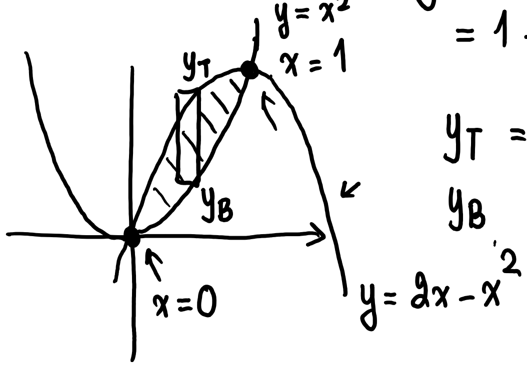
$$A = \int_0 (e^x - x)\,dx = \left[e^x - \frac{x^2}{2} \right]_0$$

$$= \left[e^1 - \frac{1}{2} \right] - \left[e^0 - 0 \right]$$

$$= e^1 - \frac{1}{2} - 1 = e - \frac{3}{2}$$

Ex Find the area bounded by the curves $y = x^2$ and $y = 2x - x^2$.

$$= 1 - (x-1)^2$$



$$y_T = 2x - x^2$$
$$y_B = x^2$$

We first need to find the points of intersection by solving the two equations simultaneously.

$$y = x^2 \ , \ y = 2x - x^2$$

$$2x - x^2 = x^2 \Rightarrow 0 = x^2 + x^2 - 2x$$

$$\Rightarrow 0 = 2x^2 - 2x \Rightarrow 0 = 2x(x-1)$$

$$\Rightarrow x = 0 \ , \ x = 1$$

$$A = \int^1 (y_T - y_B)\,dx = \int^1 (2x - x^2) - (x^2)\,dx$$

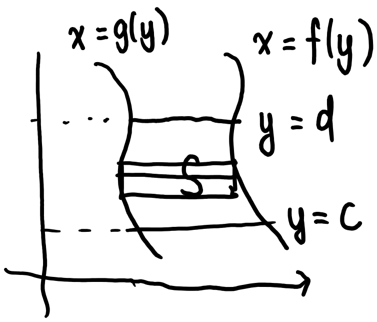
$$A = \int_0 (y_T - y_B)\,dx = \int_0 (2x - x^2) - (x^2)\,dx$$

$$= \int_0^1 2x - 2x^2\,dx = \left[x^2 - \frac{2x^3}{3} \right]_0^1$$

$$= \left[1^2 - \frac{2 \cdot 1^3}{3} \right] - \left[0 - 0 \right] = \frac{1}{3}$$

Some regions are best treated by regarding x as a function of y.

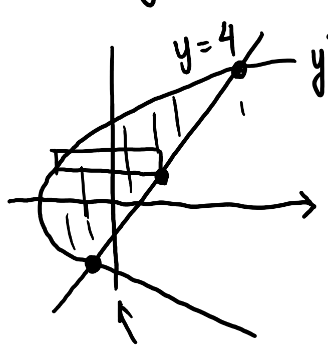
If a region S is bounded by curves $x = f(y)$, $x = g(y)$ and $y = c$, $y = d$.



$$A = \int_c^d \left[f(y) - g(y) \right] dy$$

$$= \int_c^d (x_R - x_L)\,dy$$

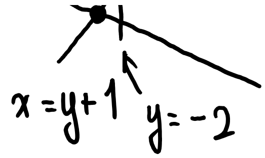
Ex Find the area enclosed by the line $x = y + 1$ and parabola $y^2 = 2x + 6$.



$$y^2 = 2x + 6 \Rightarrow 2x = y^2 - 6 \Rightarrow x = \frac{y^2 - 6}{2}$$

1) Points of intersection

$$y^2 = 2x + 6$$



$$y^2 = 2x + 6$$
$$+\ (y+1 = x)\cdot -2$$

$$y^2 - 2y - 2 = 6$$
$$\Rightarrow y^2 - 2y - 8 = 0$$
$$\Rightarrow (y-4)(y+2) = 0$$
$$\Rightarrow y = 4,\ y = -2$$

$$x_R = y + 1$$
$$x_L = \frac{y^2 - 6}{2}$$

$$A = \int_c^d x_R - x_L \, dy = \int_{-2}^{4} (y+1) - \left(\frac{y^2-6}{2}\right) dy = 18$$

D.I.Y.